

Edges of the Barvinok-Novik orbitope

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March 7, 2011

Abstract

Here we study the k th symmetric trigonometric moment curve and its convex hull, the Barvinok-Novik orbitope. In 2008, Barvinok and Novik introduce these objects and show that there is some threshold so that for two points on \mathbb{S}^1 with arclength below this threshold the line segment between their lifts to the curve form an edge on the Barvinok-Novik orbitope and for points with arclength above this threshold, their lifts do not form an edge. They also give a lower bound for this threshold and conjecture that this bound is tight. Results of Smilansky prove tightness for $k = 2$. Here we prove this conjecture for all k .

1 The odd trigonometric and cosine moment curves

Understanding the facial structure of the convex hull of curves is critical to the study of convex bodies, such as orbitopes and spectrahedron. It also reveals faces of polytopes formed by taking the convex hull of finitely many points on the curve. In 2008, Barvinok and Novik [BN] use this technique to derive new asymptotic lower bounds for the maximal face numbers of centrally symmetric polytopes. To do this they study the symmetric trigonometric moment curve and the faces of its convex hull. Following [BN], let SM_{2k} denote the symmetric trigonometric moment curve,

$$SM_{2k}(\theta) = (\cos(\theta), \cos(3\theta), \dots, \cos((2k-1)\theta), \sin(\theta), \sin(3\theta), \dots, \sin((2k-1)\theta)),$$

and B_{2k} its convex hull,

$$B_{2k} = \text{conv}(SM_{2k}([0, 2\pi])).$$

Barvinok and Novik show that B_{2k} is locally k -neighborly and use this to produce centrally symmetric polytopes with high faces numbers. The convex body B_{2k} is also an *orbitope*, that is, the convex hull of the orbit of a compact group (e.g. \mathbb{S}^1) acting linearly on a vector space, as studied in [SSS, §5]. It is also remarked that the convex hull of the full trigonometric moment curve is the Hermitian Toeplitz

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spectrahedron, meaning that B_{2k} is the projection of this Toeplitz spectrahedron [SSS]. For example,

$$B_4 = \left\{ (x_1, x_3, y_1, y_3) \in \mathbb{R}^4 : \exists z_2 \in \mathbb{C} \text{ with } \begin{bmatrix} 1 & z_1 & z_2 & z_3 \\ \bar{z}_1 & 1 & z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 & 1 & z_1 \\ \bar{z}_3 & \bar{z}_2 & \bar{z}_1 & 1 \end{bmatrix} \succeq 0 \right\}$$

where $z_j = x_j + iy_j$ and “ $M \succeq 0$ ” denotes that the Hermitian matrix M is positive semidefinite. Smilansky [S] studies in depth the convex hulls of four-dimensional moment curves, such as B_4 , and completely characterizes their facial structure.

As an orbitope, the projection of a spectrahedron, and convex hull of a curve, the centrally symmetric convex body B_{2k} is an interesting object in its own right, in addition to its ability to provide centrally symmetric polytopes with many faces. The theorem of this paper is a complete characterization of the edges of B_{2k} , which gives an affirmative answer to the first question of [BN, Section 7.4].

Theorem 1. *For $\alpha \neq \beta \in [0, 2\pi]$, the line segment $[SM_{2k}(\alpha), SM_{2k}(\beta)]$ is*

$$\begin{array}{ll} \text{an exposed edge of } B_{2k} & \text{if } |\alpha - \beta| < 2\pi(k-1)/(2k-1), \text{ and} \\ \text{not an edge of } B_{2k} & \text{if } |\alpha - \beta| > 2\pi(k-1)/(2k-1), \end{array}$$

where $|\alpha - \beta|$ is the length of the arc between $e^{i\alpha}$ and $e^{i\beta}$ on \mathbb{S}^1 .

Our contribution is to prove the second case, when $[SM_{2k}(\alpha), SM_{2k}(\beta)]$ is not an edge. The existence of exposed edges is given by the following:

Theorem 2 ([BN, Theorem 1.1]). *For all $k \in \mathbb{Z}_{>0}$, there exists $\frac{2\pi(k-1)}{2k-1} \leq \psi_k \leq \pi$ so that for all $\alpha \neq \beta \in [0, 2\pi]$, the line segment $[SM_{2k}(\alpha), SM_{2k}(\beta)]$ is an exposed edge of B_{2k} if $|\alpha - \beta| < \psi_k$ and not an edge of B_{2k} if $|\alpha - \beta| > \psi_k$.*

To prove Theorem 1, it suffices to show that for arbitrarily small $\epsilon > 0$ and $|\alpha - \beta| = 2\pi(k-1)/(2k-1) + \epsilon$, the line segment $[SM_{2k}(\alpha), SM_{2k}(\beta)]$ is not an edge of B_{2k} . By the \mathbb{S}^1 action on B_{2k} , $[SM_{2k}(\alpha), SM_{2k}(\beta)]$ is an edge of B_{2k} if and only if $[SM_{2k}(\alpha + \tau), SM_{2k}(\beta + \tau)]$ is an edge for all $\tau \in [0, 2\pi]$. Thus it suffices to show that $[SM_{2k}(-\theta), SM_{2k}(\theta)]$ is not an edge of B_{2k} for $\theta = \pi(k-1)/(2k-1) + \epsilon/2$.

To study SM_{2k} we will look at the projection onto its “cosine components”. Let

$$C_k(\theta) = (\cos(\theta), \cos(3\theta), \dots, \cos((2k-1)\theta)) \in \mathbb{R}^k.$$

By (1) below, C_k is the curve of midpoints of the line segments $[SM_{2k}(-\theta), SM_{2k}(\theta)]$.

Lemma 3. *If $C_k(\theta)$ lies in the interior of $\text{conv}(C_k)$, then $[SM_{2k}(-\theta), SM_{2k}(\theta)]$ is not an edge of B_{2k} .*

Proof. Let $L = \{x \in \mathbb{R}^{2k} : x_{k+1} = \dots = x_{2k} = 0\}$. Note that for all $\theta \in [0, 2\pi]$, $L \cap B_{2k}$ contains the point

$$(C_k(\theta), \bar{0}) = \frac{1}{2}SM_{2k}(-\theta) + \frac{1}{2}SM_{2k}(\theta), \quad (1)$$

and the convex hull of these points is full-dimensional in L . As L contains the point $(0, \dots, 0)$, it intersects the interior of B_{2k} . Thus the relative interior of $B_{2k} \cap L$ and the intersection of L with the interior of B_{2k} coincide.

By assumption, $C_k(\theta)$ lies in the interior of $\text{conv}(C_k)$, meaning that the point $\frac{1}{2}SM_{2k}(-\theta) + \frac{1}{2}SM_{2k}(\theta)$ lies in the relative interior of $L \cap B_{2k}$. Thus the line segment $[SM_{2k}(-\theta), SM_{2k}(\theta)]$ intersects the interior of B_{2k} and it cannot be an edge. \square

To prove Theorem 1, it now suffices to show that for small enough $\epsilon > 0$, $C_k(\frac{k-1}{2k-1}\pi + \epsilon)$ lies in the interior of $\text{conv}(C_k)$. It will be worth noting that $\cos(d\theta)$ is a polynomial of degree d in $\cos(\theta)$, called the d th Chebyshev polynomial [R]. Thus C_k is a segment of an algebraic curve of degree $2k - 1$, parametrized by the Chebyshev polynomials of odd degree evaluated in $[-1, 1]$.

2 Curves dipping behind facets

Here we give a criterion for a curve C to dip inside of its convex hull after meeting a facet of $\text{conv}(C)$. Let $C(t) = (C^1(t), \dots, C^n(t))$, $t \in [-1, 1]$ be a curve in \mathbb{R}^n where $C^i \in \mathbb{R}[t]$. Let F be a facet of $\text{conv}(C)$ with supporting hyperplane $\{h^T x = h_0\}$. Suppose $C(t_0)$ is a vertex of F with $t_0 \in (-1, 1)$ and that C is smooth this point (*i.e.* $C'(t_0) \neq \bar{0}$). Let π_F denote the projection of \mathbb{R}^n on to the affine span of F . See Figure 1 for an example.

Lemma 4. *If $\pi_F(C(t_0 + \epsilon))$ lies in the relative interior of F for small enough $\epsilon > 0$ and any facet of F containing $C(t_0)$ meets the curve $\pi_F(C)$ transversely at this point, then $C(t_0 + \epsilon)$ lies in the interior of $\text{conv}(C)$.*

Proof. Let p be a point on $C \setminus F$. Then $\text{conv}(F \cup p)$ is a pyramid over the facet F . We will show that $C(t_0 + \epsilon)$ lies in the interior of this polytope. Suppose $\{h^T x \leq h_0, a_i^T x \leq b_i, i = 1, \dots, s\}$ is a minimal facet description of $\text{conv}(F \cup p)$ with $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. Then $a_i^T x < b_i$ for all x in the relative interior of F .

The polynomial $h_0 - h^T C(t) \in \mathbb{R}[t]$ is non-negative for all $t \in [-1, 1]$. As this polynomial is non-zero, it has only finitely many roots. Thus, for small enough $\epsilon > 0$, $h^T C(t_0 + \epsilon) < h_0$.

Now we show that $a_i^T C(t_0 + \epsilon) < b_i$. As $h_0 - h^T C(t)$ is non-negative and zero at $t_0 \in (-1, 1)$, it must have a double root at t_0 . This implies that $h^T C'(t_0) = 0$, and thus, for any ϵ , the point $C(t_0) + \epsilon C'(t_0)$ lies in the affine span of F . As $C(t_0)$ and $C(t_0) + \epsilon C'(t_0)$ both lie in the affine span of F , we have that

$$a_i^T C(t_0 + \epsilon) = a_i^T C(t_0) + \epsilon a_i^T C'(t_0) + O(\epsilon^2), \quad \text{and} \quad (2)$$

$$a_i^T \pi_F(C(t_0 + \epsilon)) = a_i^T C(t_0) + \epsilon a_i^T C'(t_0) + O(\epsilon^2). \quad (3)$$

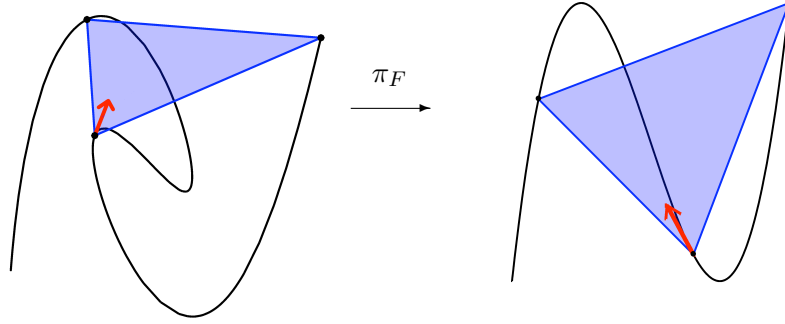


Figure 1: Projection of the curve C_3 onto the facet $\{x_3 = 1\}$ of its convex hull. The tangent vector $C_3(t_0) + C'_3(t_0)$ for $t_0 = 2\pi/5$ is shown in red.

Our transversality assumption implies that, for each $i = 1, \dots, s$, if $a_i^T C(t_0) = b_i$ then $a_i^T \pi_F(C'(t_0)) = a_i^T C'(t_0) \neq 0$. Then for small enough $\epsilon > 0$, $a_i^T C(t_0) + \epsilon a_i^T C'(t_0)$ is non-zero. As $\pi_F(C(t_0 + \epsilon))$ lies in the relative interior of F , $a_i^T \pi_F C(t_0 + \epsilon) < b_i$. By (3), this implies that $a_i^T C(t_0) + \epsilon a_i^T C'(t_0) < b_i$. It then follows from (2) that $a_i^T C(t_0 + \epsilon) < b_i$.

This shows that $C(t_0 + \epsilon)$ lies in the interior of $\text{conv}(F \cup p) \subset \text{conv}(C)$. \square

Remark 5. *The hypotheses of Lemma 4 are equivalent to the condition that for small $\epsilon > 0$, $C(t_0) + \epsilon C'(t_0)$ lies in the relative interior of F , or rather, that the vector $C'(t_0)$ lies in the relative interior of the tangent cone of F at $C(t_0)$. Given F , $C(t_0)$, and $C'(t_0)$, checking this condition is a linear program.*

3 Understanding the facet $\{x_k = 1\}$

We will show that the hypotheses of Lemma 4 are satisfied using the curve $C = C_k$, facet $F = \{x_k = 1\} \cap \text{conv}(C_k)$, and point $C(t_0) = C_k(\frac{k-1}{2k-1}\pi)$. To do this, we have to understand this facet and the projection of C_k onto the hyperplane $\{x_k = 1\}$.

Note that the intersection of C_k with the hyperplane $\{x_k = 1\}$ is k points given by solutions to $\cos((2k-1)\theta) = 1$ in $[0, \pi]$, namely $\{C_k(\frac{2j}{2k-1}\pi) : j = 0, \dots, k-1\}$. The projection of C_k onto this hyperplane is just $(C_{k-1}, 1)$. Thus to understand the projection of C_k onto this facet, we need to look at the points $\{C_{k-1}(\frac{2j}{2k-1}\pi) : j = 0, \dots, k-1\}$. Let

$$\theta_0 = \frac{\pi}{2} \quad \text{and} \quad \theta_j = \frac{2j}{2k-1}\pi \quad \text{for } j = 1, \dots, k-1.$$

Define the following two polytopes (simplices) in \mathbb{R}^{k-1} :

$$P_k = \text{conv}(\{C_{k-1}(0\pi)\} \cup \{C_{k-1}(\theta_j) : j = 1, \dots, k-1\})$$

$$Q_k = \text{conv}(\{C_{k-1}(\theta_j) : j = 0, \dots, k-1\}).$$

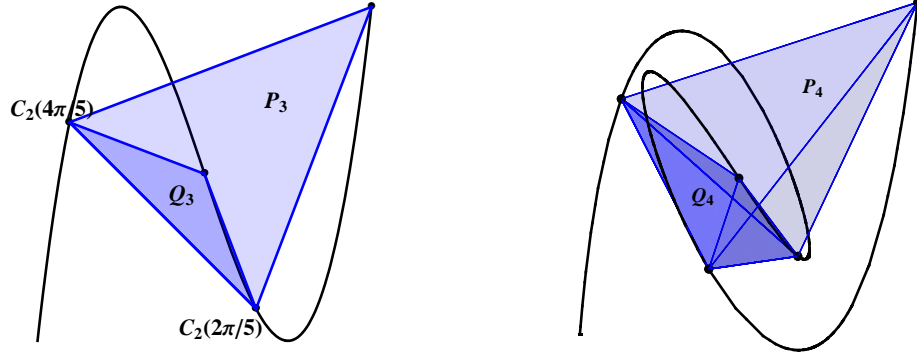


Figure 2: On the left, the curve C_2 (the projection of C_3 onto the plane $\{x_3 = 1\}$) with the triangles P_3 and Q_3 . On the right, C_3 with the tetrahedra P_4 and Q_4 .

While P_k is the polytope we'll use as F in Lemma 4, Q_k is a simplex which sits inside of P_k and has a more tractable facet description. We will show that $C_{k-1}(\frac{k-1}{2k-1}\pi + \epsilon)$ lies in Q_k in order to show that it lies in P_k . We'll often need the trigonometric identities stated in Section 5.

To see that $Q_k \subseteq P_k$, note that their vertex sets differ by only one element. It suffices to write Q_k 's extra vertex, $(0, \dots, 0) = C_{k-1}(\frac{\pi}{2})$, as a convex combination of the vertices of P_k . By Trig. Identity 1, we have that for each $l = 1, \dots, k-1$, $0 = 1/2 + \sum_{j=1}^{k-1} \cos((2l-1)\theta_j)$. Putting these together gives that $C_{k-1}(\frac{\pi}{2}) = (0, \dots, 0) = \frac{2}{2k-1}(\frac{1}{2}C_{k-1}(0\pi) + \sum_{j=1}^{k-1} C_{k-1}(\theta_j))$. So indeed $Q_k \subset P_k$.

Lemma 6. *The curve C_{k-1} meets each facet of Q_k transversely and $C_{k-1}(\theta)$ lies in the interior of $Q_k \subset P_k$ for $\theta \in \begin{cases} (\frac{(k-1)\pi}{2k-1}, \frac{\pi}{2}) & \text{if } k \text{ is odd} \\ (\frac{\pi}{2}, \frac{k\pi}{2k-1}) & \text{if } k \text{ is even.} \end{cases}$*

Proof. The plan is to find a halfspace description of Q_k , find the places where C_{k-1} crosses the boundary of each of these halfspaces, and deduce from this that $C_{k-1}(\theta)$ lies in each of these halfspaces for the appropriate θ .

First we find the facet description of Q_k . For $k \in \mathbb{N}$, and $j \in \{0, \dots, k-1\}$, define the affine linear functions $h_{j,k} : \mathbb{R}^{k-1} \rightarrow \mathbb{R}$ as

$$h_{0,k}(x) = 1/2 + \sum_{l=1}^{k-1} x_l, \quad \text{and}$$

$$h_{j,k}(x) = \sum_{l=1}^{k-1} (\cos((2l-1)\theta_j) - 1) x_l \quad \text{for } j = 1, \dots, k-1.$$

We will see that $Q_k = \{x \in \mathbb{R}^{k-1} : h_{j,k}(x) \geq 0 \text{ for all } j = 0, \dots, k-1\}$. Note that each $h_{j,k}$ gives a trigonometric polynomial by composition with C_{k-1} . For each $j = 0, \dots, k-1$, define $f_{j,k} : [0, 2\pi] \rightarrow \mathbb{R}$ by

$$f_{j,k}(\theta) := h_{j,k}(C_{k-1}(\theta)).$$

To see that the $h_{j,k}$ give a facet description of Q_k we will show that for each $j = 0, \dots, k-1$, we have $f_{j,k}(\theta_j) > 0$ and $f_{j,k}(\theta_i) = 0$ for all $i \neq j$. By Trig. Identity 2 in Section 5,

$$f_{0,k}(\theta_j) = \frac{1}{2} + \sum_{l=1}^{k-1} \cos((2l-1)\theta_j) = 0$$

for $j = 1, \dots, k-1$. Moreover $f_{0,k}(\theta_0) = f_{0,k}(\frac{\pi}{2}) = 1/2 + \sum_{l=1}^{k-1} 0 > 0$.

Now let $j \in \{1, \dots, k-1\}$. Using Trig. Identities 2 and 3, we see that for every $i \in \{1, \dots, k-1\} \setminus \{j\}$,

$$\begin{aligned} f_{j,k}(\theta_i) &= \sum_{l=1}^{k-1} \cos((2l-1)\theta_j) \cos((2l-1)\theta_i) - \sum_{l=1}^{k-1} \cos((2l-1)\theta_i) \\ &= -\frac{1}{2} - (-\frac{1}{2}) = 0. \end{aligned}$$

Also, we have $f_{j,k}(\theta_0) = f_{j,k}(\frac{\pi}{2}) = h_{j,k}(\bar{0}) = 0$. Finally

$$\begin{aligned} f_{j,k}(\theta_j) &= \sum_{l=1}^{k-1} \cos((2l-1)\theta_j)^2 - \sum_{l=1}^{k-1} \cos((2l-1)\theta_j) \\ &= \sum_{l=1}^{k-1} \cos((2l-1)\theta_j)^2 + \frac{1}{2} \quad (\text{by Trig. Identity 2}) \\ &> 0. \end{aligned}$$

So indeed $Q_k = \{x \in \mathbb{R}^{k-1} : h_{j,k}(x) \geq 0 \text{ for all } j = 0, \dots, k-1\}$.

To prove Lemma 6, it suffices to show that all roots of $f_{j,k}$ have multiplicity one and $f_{j,k}(\theta) > 0$ for the specified θ . We start by finding all roots of $f_{j,k}(\theta)$ in $[0, \pi]$.

Remark 7. As C_d is an algebraic curve of degree $2d-1$ in $\cos(\theta)$, it meets any hyperplane in at most $2d-1$ points (counted with multiplicity).

Thus for each j , $f_{j,k}$ has at most $2k-3$ roots in $[0, \pi]$. We have already found $k-1$ roots of each, namely $\{\theta_0, \dots, \theta_{k-1}\} \setminus \{\theta_j\}$. Now we find the remaining $k-2$.

(j=0). Note that $\cos(\pi-\theta) = -\cos(\theta)$. Then by Trig. Identity 2, for $i = 1, \dots, k-2$,

$$\begin{aligned} f_{0,k}\left(\frac{2i-1}{2k-3}\pi\right) &= \sum_{l=1}^{k-1} \cos\left(\frac{(2l-1)(2i-1)}{2k-3}\pi\right) + \frac{1}{2} \\ &= -1 + \sum_{l=1}^{k-2} \cos\left(\frac{(2l-1)(2i-1)}{2k-3}\pi\right) + \frac{1}{2} = -1 + \frac{1}{2} + \frac{1}{2} = 0. \end{aligned}$$

Thus the roots of $f_{0,k}$ are $\{\theta_i : i = 1, \dots, k-1\} \cup \{\frac{(2i-1)\pi}{2k-3} : i = 1, \dots, k-2\}$. As there are $2k-3$ of them, we know that these are all the roots of $f_{0,k}$ and each occurs with multiplicity one. Furthermore, since

$$\frac{k-2}{2k-3} < \frac{k-1}{2k-1} < \frac{k}{2k-1} < \frac{k-1}{2k-3},$$

it follows that $f_{0,k}$ has no roots in the interval $(\frac{(k-1)\pi}{2k-1}, \frac{k\pi}{2k-1})$. Thus the sign of $f_{0,k}$ is constant on $(\frac{(k-1)\pi}{2k-1}, \frac{k\pi}{2k-1})$. Since $f_{0,k}(\frac{\pi}{2}) > 0$, we see that for all $\theta \in (\frac{(k-1)\pi}{2k-1}, \frac{k\pi}{2k-1})$, $f_{0,k}(\theta) = h_{0,k}(C_{k-1}(\theta)) > 0$.

(j = 1, ..., k-1). Note that $f_{j,k}(\pi - \theta) = -f_{j,k}(\theta)$. We've already seen that $\theta_i = \frac{2i\pi}{2k-1}$ is a root of this function for $i \in \{1, \dots, k-1\} \setminus \{j\}$, so for each such i , $\frac{(2k-1-2i)\pi}{2k-1}$ is also a root. Thus the $2k-3$ roots of $f_{j,k}(\theta)$ are

$$\left\{\frac{\pi}{2}\right\} \cup \left\{\frac{i\pi}{2k-1} : i \in \{1, \dots, 2k-2\} \setminus \{2j, 2k-1-2j\}\right\}.$$

For each j this gives that $f_{j,k}$ has $k-1$ roots of multiplicity one in $[0, \frac{(k-1)\pi}{2k-1}]$ and no roots in $(\frac{(k-1)\pi}{2k-1}, \frac{\pi}{2})$. Note that $f_{j,k}(0\pi) < 0$. The sign of $f_{j,k}(\theta)$ changes at each of its roots, so for $\theta \in (\frac{(k-1)\pi}{2k-1}, \frac{\pi}{2})$, we have that $(-1)^{k-1} f_{j,k}(\theta) > 0$. By symmetry of $f_{j,k}(\theta)$ over $\pi/2$, we see that for $\theta \in (\frac{\pi}{2}, \frac{k\pi}{2k-1})$ we have $(-1)^k f_{j,k}(\theta) > 0$. \square

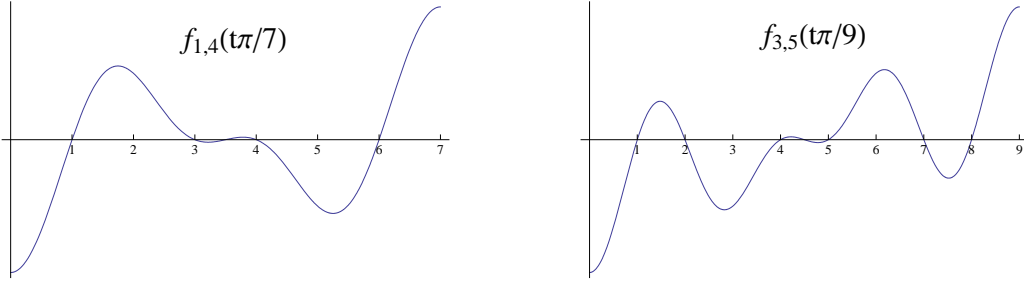


Figure 3: Here are two examples of the graphs of $f_{j,k}(\theta)$. Note that $f_{j,k}(\frac{\pi}{2k-1}t)$ has roots $\{1, \dots, 2k-1\} \setminus \{2j, 2k-1-2j\}$, all of multiplicity one.

Now that we completely understand the facets of Q_k and their intersection with the curve C_{k-1} , we can use the previous lemmata to prove our main theorem.

4 Proof of Theorem 1

Proof. As discussed before, by [BN, Thm 1.1] and symmetry of the faces it suffices to show that for arbitrarily small $\epsilon > 0$ and $\theta = \frac{k-1}{2k-1}\pi + \epsilon$, $[SM_{2k}(-\theta), SM_{2k}(\theta)]$ is

not an edge of B_{2k} . By Lemma 3, we can do this by showing that $C_k(\frac{k-1}{2k-1}\pi + \epsilon)$ lies in the interior of $\text{conv}(C_k)$.

Note that $C_k(\frac{k-1}{2k-1}\pi + \epsilon)$ lies in the interior of $\text{conv}(C_k)$ if and only if $C_k(\frac{k}{2k-1}\pi - \epsilon)$ lies in the interior of $\text{conv}(C_k)$. As the value of $\cos((k-1)\pi)$ depends on the parity of k , we will use $C_k(\frac{k-1}{2k-1}\pi + \epsilon)$ for odd k and $C_k(\frac{k}{2k-1}\pi - \epsilon)$ for even k .

We know that $\text{conv}(C_k)$ has a face given by $x_k = 1$. This intersects C_k at the points $\{C_k(0\pi)\} \cup \{C_k(\theta_j) : j = 1, \dots, k-1\}$. Thus, the intersection of $\text{conv} C_k$ with $\{x_k = 1\}$ is P_k as defined earlier sitting at height 1, and the projection of C_k onto $\{x_k = 1\}$ is C_{k-1} .

k odd. Since $k-1$ is even, $C_k(\frac{k-1}{2k-1}\pi)$ lies on the face defined by $x_k = 1$. Moreover, for small enough $\epsilon > 0$, $C_{k-1}(\frac{k-1}{2k-1}\pi + \epsilon)$ is in the interior of $Q_k \subset P_k$ by Lemma 6. As the curve C_{k-1} meets the facets of Q_k transversely at $C_{k-1}(\frac{k-1}{2k-1}\pi)$, it must meet the facets of P_k transversely at this point as well (see Remark 5). Lemma 4 then shows that $C_k(\frac{k-1}{2k-1}\pi + \epsilon)$ lies in the interior of $\text{conv}(C_k)$ for small enough $\epsilon > 0$.

k even. Now k is even and $C_k(\frac{k}{2k-1}\pi)$ lies on the face defined by $x_k = 1$. As before, for small enough $\epsilon > 0$, $C_{k-1}(\frac{k}{2k-1}\pi - \epsilon)$ is in the interior of P_k and C_{k-1} meets the facets of P_k transversely at $C_{k-1}(\frac{k}{2k-1}\pi)$. Thus $C_k(\frac{k}{2k-1}\pi - \epsilon)$ lies in the interior of $\text{conv}(C_k)$ for small enough $\epsilon > 0$. \square

We now know all the edges of B_{2k} . This leaves the challenging open problem of understanding the higher dimensional faces of this convex body.

5 Useful trigonometric identities

Trig. Identity 1. For any $k \in \mathbb{N}$ and $l \in \{1, \dots, k-1\}$,

$$\sum_{j=1}^{k-1} \cos\left(\frac{(2l-1)2j}{2k-1}\pi\right) = -\frac{1}{2}.$$

Proof. By [R, Ex. 1.5.26], for $l = 1, \dots, k-1$, we have that $0 = 1 + \sum_{j=1}^{2k-2} \cos\left(\frac{(2l-1)j}{2k-1}\pi\right)$.

As $-j \equiv 2k-1-j \pmod{2k-1}$ and $\cos(\theta) = \cos(-\theta)$, this gives

$$\begin{aligned}
0 &= 1 + \sum_{j=1}^{2k-2} \cos\left(\frac{(2l-1)j}{2k-1}2\pi\right) \\
&= 1 + \sum_{j=1}^{k-1} \left[\cos\left(\frac{(2l-1)j}{2k-1}2\pi\right) + \cos\left(\frac{(2l-1)(2k-1-j)}{2k-1}2\pi\right) \right] \\
&= 1 + 2 \sum_{j=1}^{k-1} \cos\left(\frac{(2l-1)j}{2k-1}2\pi\right).
\end{aligned}$$

□

Trig. Identity 2. For any $k \in \mathbb{N}$ and $j \in \{1, \dots, 2k-2\}$,

$$\sum_{l=1}^{k-1} \cos\left(\frac{(2l-1)2j}{2k-1}\pi\right) = -\frac{1}{2}.$$

Proof. By [R, Ex. 1.5.26], we have that for $j = 1, \dots, 2k-2$,

$$0 = \sum_{l=1}^{2k-1} \cos\left(\frac{(2l-1)2j}{2k-1}\pi\right) = 1 + \sum_{l=1}^{2k-2} \cos\left(\frac{(2l-1)2j}{2k-1}\pi\right).$$

From this, the claim follows by an argument similar to the proof of Trig. Identity 1. □

Trig. Identity 3. For any $k \in \mathbb{N}$ and $i \neq j \in \{0, \dots, k-1\}$,

$$\sum_{l=1}^{k-1} \cos\left(\frac{(2l-1)2i}{2k-1}\pi\right) \cos\left(\frac{(2l-1)2j}{2k-1}\pi\right) = -\frac{1}{2}.$$

Proof. As $|i-j|, |i+j| \in \{1, \dots, 2k-2\}$, this follows from Trig. Identity 2 and the identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$. □

Acknowledgements

Thanks to Ming Xiao Li and Raman Sanyal for many helpful discussions and to the reviewers for their careful reading. The author was funded by the University of California - Berkeley Mentored Research Award and NSF grant DMS-0757207.

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